Simple Calculations of Spin Angular Momentum

Robert A. Close
(Dated: October 16, 2024)

In quantum mechanics, students learn that angular momentum has two parts: intrinsic (or spin), and wave (or orbital) contributions. This separation is analogous to the separation of momentum into two parts when analyzing waves: intrinsic momentum associated with motion of the inertial medium, and wave momentum associated with propagation of energy by the wave. However, spin angular momentum can seem mysterious to students because, unlike the moment of momentum, it is independent of any coordinate origin. This difficulty can be overcome by teaching undergraduate students the coordinate-independent definition of intrinsic angular momentum density: the vector field whose curl is equal to twice the intrinsic momentum density. This definition of intrinsic angular momentum density, or spin density, is applicable in both classical and quantum physics. This paper gives specific examples illustrating how spin density can describe the angular momentum of rigid objects. The relationships between spin density, velocity, angular velocity, and kinetic energy are similar to the relationships between vector potential, magnetic field, electric current, and magnetic energy in magnetostatics. Appreciation of the coordinate-independent description of angular momentum will remove one obstacle to students' understanding of quantum mechanics.

Keywords: angular momentum, incompressible motion, intrinsic momentum, magnetostatics, rotational dynamics, spin, spin density

1. INTRODUCTION

Students are routinely taught that angular momentum ($\mathbf{L}$) is calculated as the "moment of momentum" $\mathbf{L} = \mathbf{r} \times \mathbf{P}$, where $\mathbf{r}$ is a position vector in some coordinate system and $\mathbf{P} = M\mathbf{u}$ is the momentum of an object with mass M and velocity $\mathbf{u}$. For objects with arbitrary composition and volume, the total angular momentum is the integral of an angular momentum density $\mathrm{L} = \mathbf{r} \times \mathbf{p}$, where $\mathbf{p} = \rho\mathbf{u}(\mathbf{r})$ is the momentum density at position $\mathbf{r}$ and ρ is the mass density. This construction interprets angular momentum as dependent on the coordinate system, making it problematic to be regarded as a fundamental physical quantity.

In contrast, the intrinsic spin angular momentum of elementary particles as defined in relativistic quantum mechanics is independent of coordinates. Belinfante [1] and Rosenfeld [2] independently demonstrated that the symmetric stress-energy tensor of general relativity requires the existence of a quantum mechanical intrinsic momentum density, which can be simplified to the vector form [3]:

$$\mathbf{p} = \frac{1}{2}\nabla \times \mathbf{s}\,, \tag{1}$$

where $\mathbf{s}$ is the density of spin angular momentum. This relationship between spin and momentum densities also has widespread application in classical physics [4–8]. In particular, this relationship is applicable to an inertial medium such as an elastic solid undergoing incompressible motion. The intrinsic and wave contributions to angular momentum for shear waves in an elastic solid are equivalent to those of relativistic quantum mechanics [9]. Therefore an understanding of classical spin angular momentum is relevant for understanding quantum mechanics as well.

Equation 1 is applicable to any physical situation as long as it is understood that $\mathbf{p}$ represents only the incompressible part of the momentum density. Yet despite its universal relevance in physics, this equation cannot be found in standard physics textbooks and is not a standard part of physics education at any level. Instead, students are incorrectly told that spin angular momentum "has no classical counterpart" [10]. While it is common in fluid mechanics to describe an incompressible velocity field as the curl of a "velocity vector potential," nowhere is it explained that this vector potential is proportional to angular momentum density. And while students commonly learn that waves on a string have both intrinsic and wave momenta, they are not taught that waves in a solid have both intrinsic (spin) and wave (orbital) angular momenta corresponding to those in quantum mechanics. This paper is evidently the first to propose that classical spin angular momentum be part of an undergraduate physics education.

In this paper, we calculate spin angular momentum for three simple physical examples with azimuthal symmetry: (1) a cylinder rotating about its axis, (2) a hollow cylinder rotating about its axis, and (3) a cylinder translating along its axis. Symmetry with respect to rotation about the axis simplifies the mathematical descriptions so that relationships between physical quantities can be easily understood.

2. METHODS

The total spin angular momentum is obtained by integrating spin density over all space. This is normally equal to the total moment of momentum computed from $\mathbf{r} \times \mathbf{p}$ [11]:

$$\begin{aligned} &\int \mathbf{r} \times \frac{1}{2}(\nabla \times \mathbf{s})d^3r = \frac{1}{2}\int \big(\nabla(\mathbf{r}\cdot\mathbf{s}) - \mathbf{r}\cdot\nabla\mathbf{s} - \mathbf{s}\cdot\nabla\mathbf{r}\big)d^3r \\ &= \frac{1}{2}\int \big(\nabla(\mathbf{r}\cdot\mathbf{s}) - \partial_i(r_i\mathbf{s}) + \mathbf{s}(\nabla\cdot\mathbf{r}) - \mathbf{s}\cdot\nabla\mathbf{r}\big)d^3r \\ &= \int \mathbf{s}\, d^3r\,. \end{aligned} \tag{2}$$

Here, the magnitude of spin density is assumed to fall to zero sufficiently rapidly at large distances for the two total derivatives above ($\nabla(\mathbf{r}\cdot\mathbf{s})$ and $\partial_i(r_i\mathbf{s})$) not to contribute to the integral. An exception to this assumption will be discussed in Sec. 3.3.

Equation 2 demonstrates that spin angular momentum is a coordinate-independent expression for ordinary angular momentum.

The usual expression for kinetic energy density (ε_K) is:

$$\varepsilon_K = \frac{1}{2}\rho u^2\,, \tag{3}$$

In terms of rotational variables, the kinetic energy density (ε_R) is:

$$\varepsilon_R = \frac{1}{2}\mathbf{w}\cdot\mathbf{s}\,, \tag{4}$$

where $\mathbf{w} = (1/2)\nabla \times \mathbf{u}$ is the angular velocity.

The total kinetic energy is [11]:

$$K = \int \frac{1}{2}\mathbf{w} \cdot \mathbf{s}\, d^3r = \int \frac{1}{2}\rho u^2 d^3r \,. \tag{5}$$

This equality follows from the vector identity:

$$\nabla \cdot (\mathbf{u} \times \mathbf{s}) = \mathbf{s} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{s}) \tag{6}$$

since the volume integral of the divergence term can be converted to a surface integral that is assumed to vanish.

According to Eq. 5, spin density ($\mathbf{s}$) is the momentum conjugate to angular velocity for any Lagrangian whose dependence on velocity is only in the kinetic energy term:

$$\frac{\delta}{\delta w_i} \int \frac{1}{2} w_j s_j \, dV = \frac{1}{2} \int \left(\frac{\delta w_j}{\delta w_i} s_j + w_j \frac{\delta s_j}{\delta w_i} \right) dV = \frac{1}{2} s_i + \frac{1}{2} s_i = s_i \,, \tag{7}$$

where integration by parts was used twice to evaluate the second term in the integral.

Since $\mathbf{p} = (1/2)\nabla \times \mathbf{s}$, we can apply Stokes' theorem to obtain:

$$\oint \mathbf{s} \cdot d\boldsymbol{\ell} = 2 \oiint \mathbf{p} \cdot \hat{\mathbf{n}} dS \,. \tag{8}$$

This relationship can sometimes simplify calculation of spin density from a known momentum density profile.

Table I shows a clear similarity between incompressible motion and magnetostatics with magnetic vector potential $\mathbf{A}$, magnetic field $\mathbf{B}$, electric current $\mathbf{J}$, and magnetostatic energy density ε_M.

TABLE I: Comparison of Magnetostatics and Incompressible Motion

Magnetostatics	**Incompressible Motion**
$\nabla \times \mathbf{A} = \mathbf{B}$	$\nabla \times \mathbf{s} = 2\rho\mathbf{u}$
$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$	$\nabla \times \mathbf{u} = 2\mathbf{w}$
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{u} = 0$
$\varepsilon_M = \frac{B^2}{2\mu_0}$	$\varepsilon_K = \frac{1}{2}\rho u^2$

Therefore an understanding of spin density could also help students understand magnetostatics. An interesting distinction between the two physical phenomena is that while total angular momentum is an important physical quantity, the volume integral of magnetic potential is not known to be a useful concept.

3. RESULTS

3.1. Rotating Cylinder

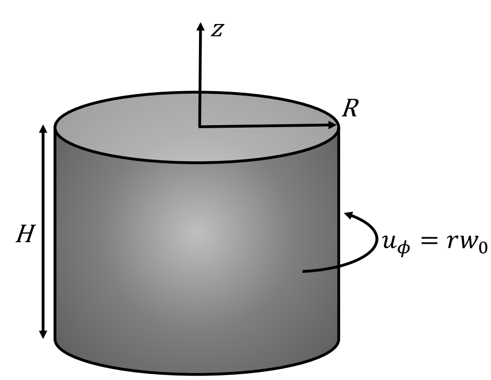

FIG. 1: A Rotating Cylinder.

We will use spin density to describe a cylinder aligned with the z-axis and rotating rigidly with angular velocity w_0 (Fig. 1). The momentum density is:

$$p_\phi = \left\{ \begin{array}{cl} \rho r w_0 & \text{for } r \le R \text{ and } 0 \le z \le H \\ 0 & \text{for } r > R \text{ or } z < 0 \text{ or } z > H \end{array} \right\} . \tag{9}$$

For $0 \le z \le H$ and $r < R$, the differential equation for $\mathbf{s}(\mathbf{r})$ is:

$$\frac{1}{4\rho} \nabla \times (\nabla \times \mathbf{s}) = w_0 \hat{\mathbf{z}} \,. \tag{10}$$

Since the right-hand side is the in z-direction, we look for a solution with $s_r = s_\phi = 0$. Azimuthal symmetry implies that the spin density satisfies the equation:

$$-\frac{1}{4\rho r} \partial_r (r \partial_r s_z) = w_0 \,. \tag{11}$$

The general solution is:

$$s_z(r) = -\rho w_0 r^2 + c_1 \ln r + c_2 \tag{12}$$

where c_1 and c_2 are arbitrary constants. Requiring a finite value at $r = 0$ implies that $c_1 = 0$, and requiring $s_z \to 0$ for $r \to R$ requires $c_2 = \rho w_0 R^2$. Therefore the solution inside the cylinder is [11]:

$$s_z(r) = \rho w_0 (R^2 - r^2) \,. \tag{13}$$

Outside the cylinder, the equation for s_z is:

$$-\frac{1}{4\rho r} \partial_r (r \partial_r s_z) = 0 \,. \tag{14}$$

The solution to this equation is $s_z(r) = c_1 \ln r + c_2$. Requiring $s_z = 0$ at $r \to \infty$ requires $c_1 = c_2 = 0$. The complete solution is therefore:

$$s_z(r) = \left\{ \begin{array}{cl} \rho w_0 (R^2 - r^2) & \text{for } r < R \text{ and } 0 < z < H \\ 0 & \text{for } r > R \text{ or } z < 0 \text{ or } z > H \end{array} \right\} . \tag{15}$$

Define the step function $\chi(x)$:

$$\chi(x) = \left\{ \begin{array}{cl} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{array} \right\} . \tag{16}$$

Then:

$$s_z(r, z) = \rho w_0 (R^2 - r^2) \big(1 - \chi(r - R)\big) \big(\chi(z) - \chi(z - H)\big) \,. \tag{17}$$

The calculated velocity is:

$$u_\phi = -\frac{1}{2\rho} \partial_r s_z = r w_0 \big(1 - \chi(r - R)\big) \big(\chi(z) - \chi(z - H)\big) \,. \tag{18}$$

The calculated components of angular velocity are:

$$w_z = \tfrac{1}{2r} \partial_r (r u_\phi) = w_0 \big(1 - \tfrac{1}{2} R \delta(r - R)\big) \big(\chi(z) - \chi(z - H)\big) \,, \tag{19a}$$

$$w_r = -\tfrac{1}{2} \partial_z (u_\phi) = -\tfrac{1}{2} r w_0 \big(1 - \chi(r - R)\big) \big(\delta(z) - \delta(z - H)\big) \,. \tag{19b}$$

Note that this satisfies $\nabla \cdot \mathbf{w} = 0$ everywhere.

An alternative derivation of the spin density utilizes Stokes' theorem. Once we established that the spin density is entirely in the z-direction inside the cylinder, Stokes' theorem can be applied to a rectangular loop with one side along the z-axis and the opposite side at radius r:

$$H \big(s_z(0) - s_z(r)\big) = H r \big(\rho r w_0\big) \,. \tag{20}$$

Solving for $s_z(r)$ yields:

$$s_z(r) = s_z(0) - \rho r^2 w_0 \,. \tag{21}$$

Requiring $s_z(R) = 0$, the value on the axis is $s_z(0) = \rho R^2 w_0$ and the value of s_z inside the cylinder is again given by Eq. 13.

The total angular momentum is:

$$S_z = \int s_z \, d^3r = \rho w_0 2\pi H \int_0^R \left(R^2 - r^2\right) r \, dr = \rho w_0 2\pi H \left(\frac{R^4}{4}\right) . \tag{22}$$

Identifying the mass as $M = \rho \pi R^2 H$, this is:

$$S_z = \frac{MR^2}{2} w_0 \,. \tag{23}$$

This is the usual expression for angular momentum of a cylinder with moment of inertia $I = MR^2/2$.

Since the angular velocity is constant within the cylinder, the kinetic energy is easily calculated:

$$K = \int \frac{1}{2} \mathbf{w} \cdot \mathbf{s} \, d^3r = \frac{1}{2} w_0 S_z = \frac{1}{2} I w_0^2 \,. \tag{24}$$

This is of course the standard expression for the kinetic energy of a rotating cylinder.

3.2. Rotating Hollow Cylinder

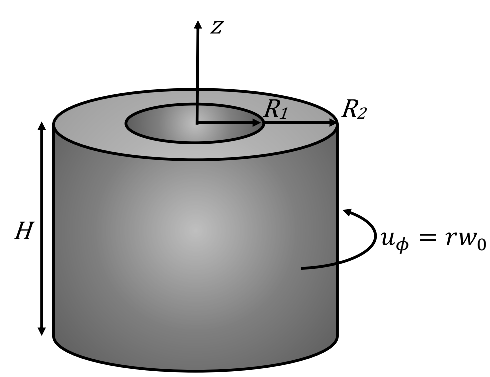

FIG. 2: A Rotating Hollow Cylinder

A rotating hollow (or annular) cylinder, as shown in Fig. 2, can be regarded as the difference between a larger cylinder with radius R_2 and a smaller cylinder with radius R_1 sharing the same rotation axis.

From Eq. 17 this yields:

$$s_z = \rho w_0 \big(\chi(z) - \chi(z-H)\big) \left\{ \left(R_2^2 - r^2\right) \big(1 - \chi(r - R_2)\big) - (R_1^2 - r^2)\big(1 - \chi(r - R_1)\big) \right\} . \tag{25}$$

Rearranging:

$$s_z = \rho w_0 \big(\chi(z) - \chi(z-H)\big) \left\{ \left(R_2^2 - R_1^2\right) + \left(R_1^2 - r^2\right) \chi(r - R_1) - (R_2^2 - r^2)\chi(r - R_2) \right\} . \tag{26}$$

This means that spin density is constant at $\rho w_0 \left(R_2^2 - R_1^2\right)$ for $r < R_1$, then becomes $\rho w_0 \left(R_2^2 - r^2\right)$ for $R_1 \leq r \leq R_2$, then drops to zero for $r > R_2$. Note that the spin density is nonlocal in the sense that it is nonzero in a region with no motion near the center of the annulus. Although this non-locality is somewhat counterintuitive, this profile does yield the correct total angular momentum:

$$\begin{aligned}\int s_z\, d^3r &= \rho w_0 2\pi H \left\{ \int_0^{R_1} \left(R_2^2 - R_1^2\right) r\, dr + \int_{R_1}^{R_2} \left(R_2^2 - r^2\right) r\, dr \right\} \\ &= \rho w_0 2\pi H \left\{ \left(R_2^2 - R_1^2\right) \tfrac{R_1^2}{2} + R_2^2 \left(\tfrac{R_2^2 - R_1^2}{2}\right) - \tfrac{R_2^4 - R_1^4}{4} \right\} \\ &= \rho w_0 2\pi H \left(\tfrac{R_2^4 - R_1^4}{4}\right) . \end{aligned} \tag{27}$$

Factoring out the mass $M = \rho\pi(R_2^2 - R_1^2)H$ yields:

$$S_z = \frac{M(R_2^2 + R_1^2)}{2} w_0 \,. \tag{28}$$

This is the usual expression for angular momentum of a hollow cylinder with moment of inertia $I = M(R_2^2 + R_1^2)/2$.

Like the solid cylinder, the angular velocity is constant within the hollow cylinder, so the total kinetic energy is again simply $K = (1/2)Iw_0^2$.

3.3. Translating Cylinder

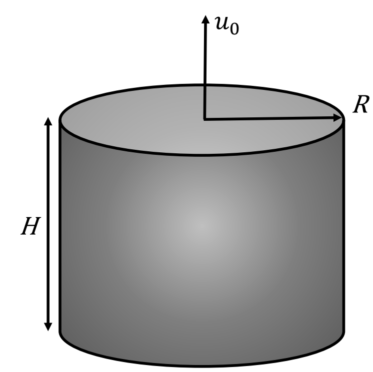

FIG. 3: A Translating Cylinder.

Our final example is a cylinder translating along the direction of its axis as in Fig. 3. In this case $\nabla \cdot \mathbf{p} \neq 0$ at the top and bottom of the cylinder, but we still assume that $\mathbf{p} = (1/2)\nabla \times \mathbf{s}$ in the region of interest. The momentum density profile is:

$$p_z = \left\{ \begin{array}{cl} \rho u_0 & \text{for } r \leq R \text{ and } 0 < z < H \\ 0 & \text{for } r > R \text{ or } z < 0 \text{ or } z > H \end{array} \right\} . \tag{29}$$

This has an angular velocity profile of:

$$w_\phi(r) = -\frac{1}{2\rho}\partial_r p_z(r,z) = \frac{1}{2} u_0 \delta(r - R)\big(\chi(z) - \chi(z - H)\big) \,. \tag{30}$$

We can use Stokes' Theorem to find the z-component of spin density. For $r < R$ and $0 < z < H$, we have:

$$\oint_0^{2\pi} s_\phi(r) r\, d\phi = \iint \rho u_0 r d\phi dr \,, \tag{31}$$

which yields:

$$s_\phi(r) = \rho u_0 r \,. \tag{32}$$

For $r > R$ and $0 < z < H$, the area integral is constant but the line integral increases with radius, so:

$$s_\phi(r) = \frac{\rho u_0 R^2}{r}\,. \tag{33}$$

Combining the different regions and specifying the range of z yields:

$$s_\phi(r,z) = \rho u_0\big(r(1-\chi(r-R)) + \frac{\rho u_0 R^2}{r}\chi(r-R)\big)\big(\chi(z)-\chi(z-H)\big)\,. \tag{34}$$

In this case the spin density falls to zero at infinity only as $1/r$, so the volume integral of spin density could depend on where the integration boundary is chosen. If the boundary is chosen to have azimuthal symmetry around the axis of the cylinder, then the spin density integrates to zero:

$$\mathbf{S} = \int_V s_\phi(r,z)\hat{\boldsymbol{\phi}}\, r d\phi\, dr\, dz = 0\,. \tag{35}$$

In this case there are always equal and opposite contributions from points separated by 180-degree rotation.

However, if the integration boundary is not symmetrical with respect to the cylinder's axis, then the cancellation is incomplete and a net integrated angular momentum would result. Consider a boundary with radius $R_B + x_0\cos\phi$ where $R_B >> R$ and $x_0 << R_B$. This approximates a displacement from the cylinder axis by $x_0\hat{\mathbf{x}}$. The integral of spin density is:

$$\begin{aligned}\mathbf{S} &= \textstyle\int_{z=0}^{H}\int_{\phi=0}^{2\pi}\int_{r=R}^{R_B+x_0\cos\phi}\frac{\rho u_0 R^2}{r}\big(\hat{\mathbf{y}}\cos\phi - \hat{\mathbf{x}}\sin\phi\big)dz r d\phi dr\\ &= H\textstyle\int_{\phi=0}^{2\pi}\rho u_0 R^2\big(\hat{\mathbf{y}}\cos\phi - \hat{\mathbf{x}}\sin\phi\big)\big(R_B + x_0\cos\phi - R\big)d\phi\\ &= H\pi\rho u_0 R^2\hat{\mathbf{y}}x_0 = M u_0 x_0\hat{\mathbf{y}}\,. \end{aligned} \tag{36}$$

This is the same result we would have gotten by integrating $(\mathbf{r} - x_0\hat{\mathbf{x}})\times\mathbf{p}$. In this case, shifting the integrated volume of spin density has the same effect as a shift of the origin for calculating moment of momentum.

This example illustrates the effect of spin density contributions at large distances from the motion (or from integration boundaries). For a rigid solid object, there is no problem limiting integration to the solid region. More generally, the total calculated spin angular momentum varies with the choice of integration boundary, but the spin density is always well-defined and independent of coordinates.

The kinetic energy for the translating cylinder is calculated to be:

$$K = \int_V \frac{1}{2}w_\phi s_\phi d^3r = \frac{1}{2}(2\pi H)\int_{r=0}^{R}\left(\frac{1}{2}u_0\delta(r-R)\right)(\rho u_0 r)\, r\, dr = \pi H\left(\frac{1}{2}\rho u_0^2 R^2\right) = \frac{1}{2}M u_0^2\,, \tag{37}$$

which is simply the kinetic energy of a moving mass.

4. DISCUSSION

These examples demonstrate the role of spin density in describing rotational motion. Far from being an exclusively quantum mechanical property, spin density is an intrinsic property of any physical system with rotational motion. Because it is independent of coordinates, spin density is a more fundamental quantity than the moment of momentum that is ordinary taught to students.

Furthermore, the physical relationship between spin density and intrinsic momentum density is the same in both classical and quantum physics. The incompressible motion of shear waves in an elastic solid is naturally described in terms of spin density. An intuitive second-order nonlinear equation for these waves can be factored using bispinors to yield a first-order Dirac equation with nonlinear terms replacing the mass term in the electron equation [9, 11]. The only reason this may seem surprising is that in quantum mechanics the units of angular momentum (in the form of $\hbar$) are factored out of physical quantities so that instead of spin density being $(1/2)\psi^\dagger\boldsymbol{\sigma}\psi$, it is written as $(\hbar/2)\psi^\dagger\boldsymbol{\sigma}\psi$ and the wave function is taken to be dimensionless. In spite of this misleading manipulation, the Dirac equation of relativistic quantum mechanics is in fact a deterministic equation that describes the evolution of spin density. It can rightly be regarded as a classical wave equation.

5. CONCLUSIONS

We have computed total angular momentum and kinetic energy for three solid objects with azimuthal symmetry. These calculations illustrate the utility of spin density as a fundamental physical quantity. Unlike the moment of momentum density, spin density is independent of coordinates and can therefore be regarded as an intrinsic property of the physical system. Since there is a close analogy between the variables of incompressible motion and the variables of magnetostatics, an understanding of spin density can help students understand the relationships between magnetostatic variables. And since the relationship between spin and intrinsic momentum densities is the same for both classical and quantum physics, an understanding of spin density will make quantum mechanics somewhat less mysterious for students.

DATA AVAILABILITY STATEMENT

No new data were generated or analyzed in this study.

REFERENCES

[1] F. J. Belinfante, Physica **6**, 887 (1939).
[2] L. Rosenfeld, Mmoires Acad. Roy. de Belgique **18**, 1 (1940).
[3] H. C. Ohanian, Am. J. Phys. **54**, 500 (1986).
[4] W. L. Jones, J. Fluid Mech **58**, 737 (1973).
[5] R. P. Cameron, S. M. Barnett, and A. M. Yao, New J. Phys **14**, 053050 (2012).
[6] K. Y. Bliokh, A. Y. Bekshaev, and F. Nori, New J. Phys. **15**, 033026 (2013).
[7] A. Aiello, P. Banzer, M. Neugebauer, and G. Leuchs, Nat. Photon. **9**, 789 (2015).
[8] M. Partanen and J. Tulkki, Proc. of SPIE **12901**, 129010F (2024).
[9] R. A. Close, in *Ether Space-time and Cosmology*, Vol. 3, edited by M. C. Duffy and J. Levy (Apeiron, Montreal, 2009) pp. 49–73.
[10] R. Shankar, *Principles of Quantum Mechanics* (Springer, New York, 1994) pp. 567–569.
[11] R. A. Close, Elect. J. Theor. Phys. **12**, 43 (2015), url = https://www.aracne-editrice.it/index.php/pubblicazione.html?item=9788854889095.